EXERCISES FOR FINITE GROUP REPRESENTATIONS

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Here are some exercises on representations of finite groups.

(1) Let $\phi\colon \mathbb{Z}_3 \to GL_2(\mathbb{C})$ be defined by
$$\phi_{[k]} := A^k, \qquad A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}.$$
Show that ϕ is not an irreducible complex representation.

(2) The formula of the previous problem actually defines a homomorphism $\phi'\colon \mathbb{Z}_3 \to GL_2(\mathbb{R})$. Show that ϕ' is an irreducible real representation (i.e., that the only invariant vector subspaces of $\mathbb{R}^2$ are 0 and $\mathbb{R}^2$).

(3) Let $\phi, \psi\colon G \to GL_1(\mathbb{C})$ be 1-dimensional representations. Show that $\phi \sim \psi$ iff $\phi = \psi$.

(4) Let $\phi\colon G \to GL(V)$ be a representation, and let $\theta\colon G \to GL(V)$ be a homomorphism. Define $V^\theta := \{\, v \in V \mid \phi_g(v) = \theta_g(v) \text{ for all } g \in G \,\}$. Show that V^θ is a G-invariant subspace of the representation ϕ.

(5) Show that if $\phi\colon G \to GL_n(\mathbb{C})$ is a representation, then so is $\psi\colon G \to GL_n(\mathbb{C})$, defined by $\psi_g := \overline{\phi_g}$, the matrix obtained by taking complex conjugates of the entries. Give an example of a ϕ so that $\phi \not\sim \psi$.

(6) Show that there is a representation $\phi\colon D_4 \to GL_3(\mathbb{C})$ with
$$\phi_r = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \qquad \phi_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$
Determine all the invariant subspaces of this representation, and show that ϕ is equivalent to a direct sum of two of its non-trivial invariant subspaces.

(7) There is a representation $\phi\colon A_4 \to GL_3(\mathbb{C})$ with
$$\phi_{(1\,2\,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \phi_{(1\,2)(3\,4)} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$
Show that ϕ is irreducible.

(8) Let $\phi\colon G \to GL(V)$ be a representation of a finite group of dimension d, with character χ. Show that if $o(g) = 2$, then $\chi(g) \in \{-d, -d+2, \dots, d-2, d\}$.

(9) Let $\phi\colon G \to GL(V)$ be a representation of a finite group of dimension d, with character χ. Show that $\|\chi(g)\| \leq d$ for any $g \in G$, with $\chi(g) = d$ iff $g \in \mathrm{Ker}(\phi)$. (Hint: use the triangle inequality for complex numbers.)

(10) Let $\phi\colon G \to GL(V)$ be an *irreducible* representation of a finite group of dimension d, with character χ. Show that if $a \in Z(G)$ (the center of G), then $\|\chi(a)\| = d$, and that $\chi(ga) = \chi(g)\chi(a)/d$ for any $g \in G$. (Hint: use Schur's lemma.)

(11) Show that there are exactly n distinct irreducible representations of $\mathbb{Z}_n$.

(12) Let G, H be finite abelian groups. Show that there is a bijection $\mathrm{Hom}(G \times H, \mathbb{C}^\times) \approx \mathrm{Hom}(G, \mathbb{C}^\times) \times \mathrm{Hom}(H, \mathbb{C}^\times)$ (where $\mathrm{Hom}(G, K)$ = the set of group homomorphisms from G to K).

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(13) Compute the character table for $\mathbb{Z}_3 \times \mathbb{Z}_3$, and give your reasoning.

(14) Compute the character table for A_4, and give your reasoning. Let $\rho' \colon A_4 \to GL_4(\mathbb{C})$ be the restriction of the regular representation of S_4 to the subgroup A_4. Determine the multiplicities of the irreducible A_4-representations in ρ'.

(15) Compute the character table for Q_8, and give your reasoning. Show that there is a representation $\phi \colon Q_8 \to GL_4(\mathbb{C})$ with

$$\phi_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \phi_j = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Determine the multiplicities of the irreducible Q_8-representations in ϕ.

(16) Explain how to deduce the character table for S_4. (The table is shown in the notes.)

(17) Explain how each irreducible S_4 representation splits into a direct sum of irreducible A_4 representations. (Use the character table for A_4 computed above.)

(18) Consider the 3 dimensional complex representation of S_4, coming from the fact that S_4 is isomorphic to the subgroup $G \leq SO(3) \leq GL_3(\mathbb{R}) \leq GL_3(\mathbb{C})$. Determine the character of this representation, and identify its decomposition as a direct sum of irreducible complex representations.

(19) Let $H \leq G$. Show that if $\chi \in L^c(H)$ is the trivial character ($\chi(h) = 1$ for all $h \in H$), then $\chi' := \operatorname{Ind}_H^G \chi$ is the character of the permutation representation $\rho \colon G \to GL(\mathbb{C}(G/H))$ of the left-coset action by G on G/H.

(20) (40 points) G is a finite group with 7 conjugacy classes. The characters $\chi_1, \dots, \chi_7$ of its irreducible complex representations are described by the following table.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1
χ_3	1	1	1	-1	-1	1	-1
χ_4	1	1	1	-1	-1	-1	1
χ_5	2	2	-2	0	0	0	0
χ_6	2	-2	0	$i\sqrt{2}$	$-i\sqrt{2}$	0	0
χ_7	2	-2	0	$-i\sqrt{2}$	$i\sqrt{2}$	0	0

Here $g_1, \dots, g_7$ are representatives of each of the conjugacy classes. Determine:

(a) The order of G.

(b) The size of each of the seven conjugacy classes of G.

(c) The dimensions of each of the irreducible representations of G.

(d) The structure of the quotient group $G/[G,G]$, where $[G,G] = \langle aba^{-1}b^{-1},\ a, b \in G\rangle$ is the commutator subgroup.

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